The symplectic structure of the spin Calogero model.

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Abstract

We compute the symplectic structure of the spin Calogero model in terms of algebro-geometric data on the associated spectral curve.

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1 Introduction

In a recent paper [1] I.M. Krichever and D.H. Phong made an important progress in the understanding of the symplectic structure of integrable models. Namely they were able to relate the canonical symplectic structure of the model with a naturally defined symplectic form on a suitable fibered space constructed with algebro-geometric data. In their paper this construction is illustrated by a number of examples. The next natural example to be treated is the spin Calogero model, and we show in this paper that the same general construction applies as well. This model has been solved in [2], where it was found that as compared to the scalar case a number of interesting new features enrich the algebro-geometric analysis.

2 The model.

2.1 Definition.

The spin Calogero model consists of N particles on a line with internal degrees of freedom. The particles are described by their positions x_i and momenta p_i, together with spin variables f_{ij}. The Hamiltonian is:

$$H = \frac{1}{2}\sum_i p_i^2 - \frac{1}{2}\sum_{i\neq j} \frac{f_{ij}f_{ji}}{(x_i - x_j)^2} \tag{2.1}$$

The non vanishing Poisson brackets for these degrees of freedom are:

$$\begin{aligned} \{p_i, x_j\} &= \delta_{ij} \\ \{f_{ij}, f_{kl}\} &= \delta_{jk} f_{il} - \delta_{il} f_{kj} \end{aligned} \tag{2.2}$$

We see that the Poisson bracket 2.2 is a Kirillov bracket and therefore is degenerate. We shall choose an orbit such that the matrix of spin variables f is of rank l.

This system admits a Lax pair formulation, with Lax operator:

$$L_{ij}(t, z) = p_i \delta_{ij} + (1 - \delta_{ij}) f_{ij} \Phi(x_i - x_j, z) \tag{2.3}$$

where for the elliptic model the special function Φ is defined in terms of Weierstrass elliptic functions by:

$$\Phi(x, z) = \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}$$

It is elliptic in z and pseudo-periodic in x. It has been shown in [3] that the model is integrable only when all f_{ii} are equal, e.g. $f_{ii} = 2$.

2.2 The spectral curve.

In the article [2] it is noticed that this model can be related to the matrix KP equation, and this leads to its solution by algebro-geometric methods. In this paper we shall be concerned with the Hamiltonian aspect of the solution. The main object of interest is the spectral curve:

$$R(k, z) \equiv \det(2kI + L(t, z)) = 0 \tag{2.4}$$

This defines a Riemann surface Γ of genus $g = Nl - l(l+1)/2 + 1$ (see [2]) which is time–independent due to the Lax equation. The equation 2.4 is polynomial in k of degree N with coefficients elliptic functions of z, hence presents Γ as an N–sheeted covering of an elliptic curve with $2g-2$ branch points. On such a curve one usually defines a line bundle by taking at each point $P = (z, k)$ the eigenspace of $L(t, z)$ for the eigenvalue k, i.e. the solutions of:

$$(2kI + L(t, z))C(t, P) = 0$$

In particular at the branch points the eigenspaces for the two colliding eigenvalues are generically of dimension 1, and $L(t, z)$ is not diagonalizable for such z.

Here special attention is required for the points above $z = 0$.

2.3 The vicinity of $z = 0$

At $z = 0$ the Lax matrix L has an essential singularity since $\zeta(z) = 1/z + O(z^3)$. Hence one can write:

$$L(z) = \mathrm{Diag}(e^{\frac{1}{z}(x_k - x_0)})\tilde{L}(z)\mathrm{Diag}^{-1}(e^{\frac{1}{z}(x_k - x_0)})$$

where x_0 is some arbitrary origin, and $\tilde{L}$ is meromorphic in a vicinity of $z = 0$. The eigenvectors are of the form $C = (C_i)$ with:

$$C_i(P) = e^{\frac{1}{z}(x_i - x_0)}\tilde{C}_i(P)$$

where $\tilde{C}_i(P)$ is a locally analytic eigenvector of $\tilde{L}$. Since $\tilde{L}(z) = -\frac{1}{z}(f - 2I) + O(1)$ we see that $\tilde{C}_i(z = 0)$ is an eigenvector of $f - 2I$ and the corresponding eigenvalues are of the form $k = (-1 + \lambda/2)/z$, where λ are the eigenvalues of f, and $N - l$ of them vanish[1]. The class of functions having essential singularities at some points of the form $\psi = \exp(\alpha/z^m)\rho(z)$ (here $m = 1$) with ρ locally meromorphic, and meromorphic otherwise are called Baker functions. The solution of many integrable models by algebro–geometric methods essentially depends on the construction of appropriate Baker functions on the spectral curve. A Baker function has properties similar to ordinary meromorphic functions, e.g. has the same number of zeroes and poles (consider the sum of residues of the regular differential $d\psi/\psi$) but theorems such as Riemann–Roch have to be modified, for example there exists non trivial Baker functions with arbitrary prescribed g poles on a surface of genus g while for a meromorphic function one needs to prescribe $g+1$ poles in the generic situation. To construct such a Baker function ψ let us remark that one can find a unique normalized abelian differential of second kind ω_2 and differential of third kind $\omega_3(\rho_k)$ depending on the given poles, and the unknown zeroes ρ_k of ψ, such that $d\psi/\psi = \omega_2 + \omega_3(\rho_k)$. By definition this form integrates to 0 on A–cycles. Imposing the same condition on B–cycles provides a system of g equations for the g unknowns ρ_k. For a construction using theta functions see [2]. Let us also remark that the quotient of two Baker functions with the same type of singularities is a meromorphic function to which one can apply the usual Riemann–Roch theorem.

More generally one can define Baker line bundles, by imposing Baker conditions on the components around singular points, and this is the case for the eigenvector bundle of

[1] Remark that this defines N generically different branches of Γ which intersect at the singular point $(z = 0, k = \infty)$. After one blow–up l different points appear, and an ordinary multiple point of order $N - l$ remains. An other blow–up at this point leaves us with N different points above $z = 0$. When we speak of Γ we have in mind this desingularization, and we speak freely of the N points P_α above $z = 0$.

L. This line bundle is of Chern class $-(g-1)$. In fact in [2] a nowhere vanishing section $C(t,P)$ is constructed with $g-1$ poles.

Since L_{ij} is not a symmetric matrix it is also convenient to construct an other Baker line bundle by considering the adjoint eigenvector equation:

$$C^+(t,P)(2kI + L(t,z)) = 0$$

which is also of Chern class $-(g-1)$. Here the Baker behaviour occurs with opposite exponent:

$$C^+(P) = e^{\frac{-1}{z}(x_i - x_0)}\tilde{C}^+(P)$$

Note that these Baker line bundles are embedded in the ambient space of dimension N, which provides a pairing:

$$< C^+(P), C(P') >= \sum_i C_i^+(P)C_i(P')$$

In particular for two points P and P' above the same z, corresponding to different eigenvalues, one has $< C^+(P), C(P') >= 0$ due to the eigenvector equations, hence at the branch points $< C^+(P), C(P) >$ vanishes for any regular sections C^+, C. Moreover the singular factors $\exp \frac{\pm 1}{z}(x_i - x_0)$ cancel above $z = 0$ so $< C^+(P), C(P) >$ extends to a meromorphic function on Γ.

Finally let P_α be the N points above the same z, and let $C_\alpha = C(P_\alpha)$ be the corresponding eigenvectors. Except at the branch points, they form a basis of ambient space, and any vector V can be decomposed as:

$$V = \sum_{\alpha=1}^{N} \frac{< C_\alpha^+, V >}{< C_\alpha^+, C_\alpha >} C_\alpha \tag{2.5}$$

2.4 Remarkable abelian forms.

The Riemann surface Γ presented by equation 2.4 possesses several remarkable abelian differentials. First the form dz is well defined on Γ and has no pole. As such it has $2g-2$ zeroes which are the branch points (where k is the local analytic parameter and dz/dk vanishes). The form of main interest is the form kdz which has poles only above $z = 0$. Finally let us take two non vanishing sections C^+, C of the above line bundles. Then $< C^+(P), C(P) >$ is a meromorphic function without any singularity above $z = 0$ vanishing at the same points as dz hence $\Omega = dz/ < C^+(P), C(P) >$ is an other analytic abelian form vanishing at the $2g-2$ poles γ_k and γ_k^+ of C and C^+. It will play an important role later on.

3 The canonical symplectic structure.

Due to the degeneracy of the Poisson brackets 2.2 one has to be careful about the choice of the symplectic variety. In fact this Poisson bracket is a Kirillov bracket for the coadjoint action of the group $GL(N)$ acting on the dual of the Lie algebra $gl(N)$ identified with itself by the invariant form $(A,B) \to \mathrm{Tr}(AB)$. The orbits are generically characterized by the eigenvalues of the matrix f which are in the center of the Poisson bracket. Here we shall consider matrices f of rank l with l different non–vanishing eigenvalues. Such an

orbit is of the form $\{g^{-1}\Lambda g | g \in GL(N)\}$ with $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0)$. The tangent space at the orbit of f is the set of matrices $U = [f, X]$ for any $X \in gl(n)$. In a basis where f is diagonal this equation reads $U_{ij} = (\lambda_i - \lambda_j)X_{ij}$, hence U_{ij} vanishes when $\lambda_i = \lambda_j$ but is otherwise arbitrary. So the dimension of the orbit is $2Nl - l^2 - l$. At a point f the symplectic form on two tangent vectors $[f, X]$ and $[f, Y]$ is given by the well-defined formula:

$$\omega_K([f, X], [f, Y]) = \mathrm{Tr}(f[X, Y])$$

in accordance with Kirillov's prescription. Explicitly in a basis where f is diagonal this can be written:

$$\omega_K = \sum_{\substack{i,j \\ \lambda_i \neq \lambda_j}} \frac{df_{ij} \wedge df_{ji}}{\lambda_i - \lambda_j} \tag{3.6}$$

where by definition on a tangent vector U, $df_{ij}(U) = U_{ij}$.

Of course the symplectic form of our model is

$$\omega = \sum dp_i \wedge dx_i + \omega_K \tag{3.7}$$

The Hamiltonian 2.1 is not invariant under the above $GL(N)$ but it is is preserved by special subgroups. First we have the discrete subgroup of permutation matrices, i.e. the Weyl group, which simply operates by permutation of the N indices i [2], and more importantly we have the group of diagonal matrices, i.e. the Cartan torus, which operates by:

$$f_{ij} \to d_i^{-1} f_{ij} d_j$$

This action preserves the Hamiltonian which only depends on $f_{ij}f_{ji}$ and all higher Hamiltonians $\mathrm{Tr}(L^n)$. The action of this toral subgroup induces a fibering of the orbits into fibers of dimension $N-1$ since multiple of the identity leave f invariant. The moment associated to this action is the collection of diagonal elements f_{ii}, that is $N-1$ non trivial moments since on the orbit the eigenvalues of f are fixed, hence so is $\mathrm{Tr}(f)$. We consider the reduced dynamical system obtained by first fixing the moments to a common value $f_{ii} = 2$, and then quotienting by the stabilizer of this moment which is the whole diagonal group. This is the integrable system as considered above.

We can now count the number of degrees of freedom. We have $2N$ degrees for the x_i, p_i, plus $2Nl - l^2 - l$ for the orbit, minus $2(N-1)$ due to the Hamiltonian reduction which ends up to a phase space of dimension $2(Nl - l(l+1)/2 + 1) = 2g$.

It is a remarkable fact that the spectral equation 2.4 is dependent on g non trivial integrals of motion. In fact it has been shown in [2] that considering only the order of the singularity at $z = 0$ and knowing that all $f_{ii} = 2$ and f is of rank l, the spectral equation depends on $g + l - 1$ parameters. Here the eigenvalues of f are in the center of the Poisson algebra, and must not be counted as dynamical variables. There are $l - 1$ independent non vanishing eigenvalues of f since $\mathrm{Tr}(f)$ has previously been fixed to $2N$. Thus we end up with exactly g action variables.

Our task is to find g other dynamical variables which will be used to construct the angle variables. It is known that such algebraically integrable systems linearize on the Jacobian of the spectral curve, hence it is natural to use g points γ_k on Γ as complementary variables. We have found above $g - 1$ poles of a non–vanishing section of the eigenvector

[2] These actions extend obviously to actions on L by similarity transformations

bundle. If one defines $C(P)$ such that $C_1(P) = 1$ as in [2], the poles of C are given by the vanishing on Γ of the first minor of the matrix $L + 2kI$. This is the algebraic equation which relates the γ_k to the dynamical variables of the system. Note that in this minor the variables x_1 and p_1 have disappeared. This corresponds to reduction by translational symmetry, which leaves a phase space of dimension $2(g-1)$.

In order to get the full phase space it is convenient to take the product of such a section with an appropriate Baker function yielding a section with g poles and a fixed zero. It suffices to choose it with given zeroes canceling the poles of C, and one more zero at a fixed point P_0 above $z = 0$. The singular behaviour above $z = 0$ is taken to be $\exp(x_0/z)$ so that we end up with $C_i(P)$ proportional to $\exp(x_i/z)$ having g dynamical poles, and similarly for C^+. Note that the poles γ_k^+ of C^+ are determined when the poles γ_k of C are given, since the abelian form $\Omega = dz/ < C^+(P), C(P) >$ is meromorphic with only singularity a double pole at P_0, hence has no residue at P_0, and one can choose the global normalization of C^+ such that:

$$\Omega = (\frac{1}{z^2} + O(1))dz \quad P \to P_0$$

But such a form is uniquely determined when one fixes g zeroes γ_k, and it has g other determined zeroes γ_k^+ (otherwise the quotient of two such forms would be a meromorphic function with g poles and g other zeroes, which is generically forbidden). We summarize the definition of C and C^+ by stating their behaviour as $z \to 0$:

$$C_i(P) = e^{\frac{x_i}{z}} c_i(P), \quad C_i^+(P) = e^{\frac{-x_i}{z}} c_i^+(P), \quad z \to 0 \tag{3.8}$$

with c_i and c_i^+ regular (and vanish when $P \to P_0$). This together with the above condition on Ω clearly determines C and C^+ up to a constant factor λ on C and $1/\lambda$ on C^+ when the γ_k are given, i.e. when the dynamical variables are given. Moreover, the $c_i(P)$ for the N points P above $z = 0$ are N eigenvectors of f (for $P = P_0$ of course we take $c_i(P)/z$ as the corresponding eigenvector), and similarly for c_i^+.

4 The action–angle variables

4.1 Some fiber bundles.

Following the ideas of [1] we first introduce some natural bundles on the moduli space of our curves, or more specifically the g–dimensional space of action variables. Let us stress that we keep the eigenvalues of f fixed throughout our discussion. First we have the bundle $\mathcal{G}$ whose fiber above a particular spectral function $R(k, z)$ is the curve Γ of equation $R = 0$. Note that the differential dz is naturally defined on the fibers independent of R. We denote δ the differential on $\mathcal{G}$, and we see that it can be splitted into a vertical part along the fiber δ_V and an horizontal part which corresponds to differentiation with z fixed δ_H. Hence if $a_i,\ i = 1, \dots, g$ are independent action variables we can write:

$$\delta = \sum_i \frac{\partial}{\partial a_i} da_i + \frac{\partial}{\partial z} dz$$

In particular the two–form $\delta(kdz) = \delta_H k \wedge dz$ is well defined on $\mathcal{G}$. Note that k has simple poles on Γ whose residues are given by the eigenvalues of f as seen above, hence

the horizontal differentiation of these residues vanish, so the above form is regular on $\mathcal{G}$. This is one of the key observations in [1].

Similarly, we define the fiber bundle $\mathcal{J}$ whose fiber above R is the Jacobian of the curve of equation $R=0$. A point of the Jacobian can be seen generically as an unordered set of g points $\gamma_i,\ i=1,\ldots,g$ of the curve Γ, and we also denote δ the differential on $\mathcal{J}$ which can be splitted as above, simply replacing the vertical part by:

$$\delta_V = \sum_i \frac{\partial}{\partial z_i} dz_i$$

where z_i is locally the z–coordinate of γ_i. It follows that we can define [1] on $\mathcal{J}$ a regular two–form $\omega = \delta(\sum_i k_i dz_i)$ which defines a symplectic structure. The notation is not accidental since we shall see that ω precisely reduces to the canonical symplectic structure when we identify $\mathcal{J}$ to the phase space of our dynamical system.

It is important to define, as in [1], forms on $\mathcal{J}$ with values abelian forms on the curve Γ associated to the corresponding base point of moduli space. In fact these forms are defined on the bundle whose fiber above a base point R is the Cartesian product of the curve Γ of equation $R=0$ and its Jacobian $\mathrm{Jac}(\Gamma)$. A generic point of such a bundle is described by g action variables a_i, a point P on Γ, and a set of g points γ_k of Γ. The differential δ acts on the a_i and the γ_k, while the differential d acts vertically along Γ on P.

We have found particularly convenient to introduce the meromorphic two–form on $\mathcal{J}$:

$$\Phi = <\delta(C^+(P)\Omega(P)) \wedge \delta L\, C(P)> \tag{4.9}$$

which due to the P dependence and the presence of $\Omega(P) = dz/ < C^+(P)C(P) >$ has values one–form on Γ. Here L is the Lax matrix depending implicitly on the a_i and γ_k, and the phase space differentiations appear in $\delta(C^+\Omega)$ and δL. Finally the brackets $<>$ contract the i–indices of C, L and C^+. The main trick [3] is to consider the sum of the residues of Φ on Γ which must vanish. But $C^+(P)\Omega(P)$ is regular at γ_k^+ for any value of the moduli and the γ_k hence so is its δ–differential. Hence the only possible singularities of Φ are located at the g points γ_k of Γ where $C(P)$ has a pole, and the N points above $z=0$ where Ω and δL are singular.

Writing these residues, we get a relation between a two–form on $\mathcal{J}$ which happens to be the previously defined algebro–geometric symplectic form, and a two–form located above $z=0$ which boils down to the canonical symplectic form of our model.

4.2 Algebro–geometric description of the canonical symplectic structure.

We can now state the result of this paper:

Proposition. *1) The canonical symplectic form 3.7 of the spin Calogero model can be written in terms of algebro–geometric variables as:*

$$\omega = \sum_{i=1}^{g} \delta k_i \wedge \delta z_i \tag{4.10}$$

[3] A similar trick allows a considerable simplification of the proof of the main theorem in [1].

where $z_i = z(\gamma_i),\ k_i = k(\gamma_i)$.
2) If we take as angle variables the Abel transform of the divisor $\sum \gamma_i$ *namely the angles defined modulo periods of* Γ*:*

$$\theta_k = \sum_i \int^{\gamma_i} \omega_k \tag{4.11}$$

where the ω_k *are a basis of regular abelian differentials dual to a basis* $\{A_k\}$ *of A–cycles of* Γ*, then the canonically conjugated variables are given by:*

$$a_l = \oint_{A_l} kdz \tag{4.12}$$

Several remarks are in order at this point. First let us note that the first statement is exactly the same type of result advocated by Sklyanin in his famous solutions of various integrable models by his method of separation of variables [4]. In fact when one takes into account the above description of the poles of C one sees that it is really the same statement. However the result is obtained here without any appeal to the R–matrix method. Moreover the second statement shows independently that the motion linearizes on the Jacobian of the spectral curve.

Then let us note that formulae of type 4.12 have already appeared at various places. For example it is clear that they play a role in the classical and semi–classical analysis of the Neumann model [5], and have been introduced more generally in [6]. In more recent analysis of quantum integrable models [7] deformations of these formulae play a central role. One may hope that direct deformation of this description leads to the quantification of integrable systems in terms of algebro–geometric concepts.

Finally, note that the proof of 1) ⇒ 2) is straightforward using a clever argument of [1]. In fact let us define a_i according to equation 4.12, they are obviously moduli coordinates on the basis of our fiber space. Now as noted above $\delta(kdz)$ defined on $\mathcal{G}$ is an analytic form on the fibers, hence can be expanded on the basis ω_k of analytic one–forms. To find the coefficients, let us compute:

$$\oint_{A_j} \frac{\partial(kdz)}{\partial a_i} = \frac{\partial a_j}{\partial a_i} = \delta_{ij}$$

so we see that $\frac{\partial(kdz)}{\partial a_k} = \omega_k$ and we have:

$$\delta(kdz) = \sum_i \delta a_i \wedge \omega_i$$

Now, taking into account the value 4.11 of the angular variables we write the symplectic form as:

$$\omega = \sum_j \delta(k_j \delta z_j) = \sum_{ij} \delta a_i \wedge \omega_i(\gamma_j) = \sum_i \delta a_i \wedge \delta\theta_i$$

which shows that the a_i are indeed canonically conjugated to the θ_i. One should note the simplicity of this derivation which essentially uses only one ingredient: the fact that the residues of the poles of kdz are killed by δ so that $\delta(kdz)$ is regular. This can be contrasted to the involved computations inherent to previous approaches on this subject.

4.3 Proof of the main result.

Let us compute the sum of the residues of the one–form Φ on Γ (which are themselves two–forms on $\mathcal{J}$). As noted above we have to look at the points γ_k's and at the points above $z=0$. In the vicinity of a pole γ_k we can write $C^+\Omega = (z-z_k)\Psi$ with Ψ regular for $z=z_k$, hence:

$$\delta(C^+\Omega) = -C^+\Omega \frac{\delta z_k}{z-z_k} + \text{regular}$$

(here $\delta z_k = \delta_V z_k$), so the residue at γ_k is:

$$\text{Res}_{\gamma_k}\Phi = -\delta_V z_k \wedge \frac{< C^+ \delta L\, C >}{< C^+ C >}(\gamma_k)$$

Note that $\delta_H(C^+\Omega)$ does not contribute to this residue because by definition $\delta_H z_k = 0$. But $C^+L = -2kC^+$ and $LC = -2kC$ hence $\delta LC + L\delta C = -2k\delta C - 2\delta k C$ which upon bracketing with C^+ yields $< C^+\delta L\, C >= -2\delta k < C^+C >$ so the final expression for the sum of the residues at the γ_j is:

$$\sum \text{Res}_{\gamma_j} = -2\sum_j \delta k_j \wedge \delta z_j = -2\omega$$

where ω is the previously defined symplectic form on $\mathcal{J}$. Writing that the sum of the residues vanishes we have:

$$2\omega = \sum_\alpha \text{Res}_{P_\alpha}\Phi \tag{4.13}$$

where the P_α's are the N points above $z=0$.

Our next task is to compute this sum and show that it indeed reduces to the canonical symplectic form on our dynamical system. Let us remark that in the vicinity of $z=0$ we have a common local parameter z for all the curves in $\mathcal{G}$, and Ω is defined by the same normalization at P_0 so $\delta\Omega$ is regular at P_0. Due to the normalization 3.8 we can write:

$$\delta C^+ = \frac{-x_i}{z}C^+ + e^{\frac{-x_i}{z}}\delta c^+$$

We treat separately the contribution of the first term. The considered expression in the vicinity of P_α takes the form:

$$-\sum_{i,j}\frac{dz}{z}\frac{< C_i^+\delta x_i \wedge \delta L_{ij} C_j >}{< C^+C >}(P)$$

Let us introduce the matrix $M_{ij} = \delta x_i \wedge \delta L_{ij}$ which only depends on $z(P)$. Summing on the N sheets we get, in view of equation 2.5[4], the simple expression $-dz/z\text{Tr}(M)$, so the residue at $z=0$ is just $\sum_i \delta p_i \wedge \delta x_i$. For the same reason the contribution proportional to $\delta\Omega$ can only pick diagonal terms in L and there is no residue.

So we are left with the remaining contribution of $e^{\frac{-x_i}{z}}\delta c^+$. Here we must examine the $1/z$ terms present in δL. First note that all factors $e^{\frac{\pm x_i}{z}}$ cancel between $e^{\frac{-x_i}{z}}\delta c^+$, δL_{ij} and C_j. Then taking into account the identity:

$$\frac{\partial}{\partial x}\phi(x,z) = \phi(x,z)[\zeta(x+z) - \zeta(x)]$$

[4] The point P_0 presents no special problem since the factor z^2 present in the numerator cancels a similar factor in the denominator.

and the expansion $\phi(x,z) = (-1/z + \zeta(x) + O(z))\exp(\zeta(z)x)$ one sees that all terms in δp_i and δx_i are regular at $z=0$, so only terms in δf_{ij} contribute to the residue. Moreover $(1-\delta_{ij})\delta f_{ij} = \delta f_{ij}$ because $\delta f_{ii} = 0$ so one can simply replace δL_{ij} by $(-1/z)\delta f_{ij}$. The residue at $z=0$ is now obvious and the expression to be computed is therefore:

$$-\sum_\alpha \frac{<\delta c_\alpha^+ \wedge \delta f c_\alpha>}{<c_\alpha^+ c_\alpha>}$$

where c_α is the value of $c(P)$ at P_α, i.e. they are the eigenvectors of f, and similarly for c^+.

To complete the calculation note that varying the equation $fc_\alpha = \lambda_\alpha c_\alpha$ keeping λ constant (recall that eigenvalues of f are fixed) one gets by contracting with c_β^+ the relation $<c_\beta^+ \delta f c_\alpha> = (\lambda_\alpha - \lambda_\beta) < c_\beta^+ \delta c_\alpha> = -(\lambda_\alpha - \lambda_\beta) < \delta c_\beta^+ c_\alpha >$. Expanding the above residue on the basis c_β in view of 2.5 one gets:

$$-\sum_{\alpha\beta} \frac{<\delta c_\alpha^+ c_\beta>}{<c_\beta^+ c_\beta>} \wedge \frac{<c_\beta^+ \delta f c_\beta>}{<c_\alpha^+ c_\alpha>}$$

in which the sum can be restricted to $\lambda_\alpha \neq \lambda_\beta$ by the previous relation. But using 2.5 this is nothing more than:

$$+\sum_{\substack{\alpha,\beta \\ \lambda_\alpha \neq \lambda_\beta}} \frac{\delta f_{\alpha\beta} \wedge \delta f_{\beta\alpha}}{\lambda_\alpha - \lambda_\beta}$$

in which we recognize the expression 3.6 of the Kirillov form ω_K. Finally we have proven that:

$$2\omega = \sum_i \delta p_i \wedge \delta x_i + \omega_K$$

5 Conclusion.

The example treated in this paper shows once more the power and the generality of the method introduced by Krichever and Phong. Even in an intricate dynamical situation, the canonical symplectic form can be expressed in terms of algebro–geometric data with the same formula as in standard cases. Due to its simplicity and generality, this formula may very well extend to the quantum domain, as simple examples already indicate.

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